Comparing Criteria for Circular Orbits in General Relativity

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We study a simple analytic solution to Einstein's field equations describing a thin spherical shell consisting of collisionless particles in circular orbit. We then apply two independent criteria for the identification of circular orbits, which have recently been used in the numerical construction of binary black hole solutions, and find that both yield equivalent results. Our calculation illustrates these two criteria in a particularly transparent framework and provides further evidence that the deviations found in those numerical binary black hole solutions are not caused by the different criteria for circular orbits.

PACS numbers: 04.25.-g, 04.25.Dm, 04.70.-s

Binary black holes are among the most promising sources of gravitational radiation for the new generation of gravitational wave detectors LIGO, VIRGO, GEO and TAMA. Motivated by the need of theoretical models for the identification and interpretation of future gravitational wave signals, several researchers have solved the constraint equations of Einstein's field equations to construct initial data describing binary black holes in quasi-circular orbit [1, 2, 3, 4, 5, 6].

Constructing such initial data requires making several choices, including the decomposition of the initial value problem and the background geometry and topology. Moreover, solving the constraint equations provides the gravitational fields for black holes with arbitrary separation and momenta, and an additional criterion has to be applied to identify circular orbits. It is not surprising that different choices lead to physically different data. While all of these different data may be correct solutions to the constraint equations of general relativity, some may be more relevant astrophysically than others, in that they better represent a binary black hole system as it arises from inspiral from large separation.

The results of Cook [1] and Baumgarte [2] (which we will jointly refer to as CB) and Grandclément, Gourgoulhon and Bonazolla [4] (hereafter GGB) differ by about a factor of two in the orbital frequency for the innermost stable circular orbit. This discrepancy raises two questions, namely which results are more relevant astrophysically and which choice in the respective approaches are responsible for the deviations. The better agreement of the GGB results with post-Newtonian results [7] suggests that these represent binary black holes in circular orbits more accurately [8]. There is also increasing evidence that the differences between CB and GGB are related to the different decompositions of the constraint equations [11, 12]. CB adopt the conformal transverse-traceless decomposition, which allows for an analytic solution of the momentum constraint [13], while GGB adopt the conformal thin-sandwich decomposition [14, 15, 16] (see also [12]). It has been demonstrated that the two decompositions may lead to physically different data [11], and it has also been suggested that the thin-sandwich decomposition together with maximal slicing may provide a more natural framework for constructing quasi-equilibrium solutions [16].

In this Brief Report we explore the effect of another difference in the approaches of CB and GGB, namely the criterion for locating circular orbits. CB adopt a turning-point method, in which circular orbits are identified with extrema of the binding energy (see eq. (27) below), while GGB identify circular orbits by equating the ADM [17] and Komar masses [18] (eq. (31)). Since the two mass definitions agree only for stationary spacetimes, this criterion is closely related to imposing a relativistic virial theorem [19].

To explore the effect of these different criteria for circular orbits in a particularly simple and transparent framework, we apply them to an analytic solution of Einstein's equations describing a thin, spherical shell of identical collisionless particles. At every point on the shell the particles move isotropically, but all with the same speed in the plane perpendicular to the radius. In an oscillating shell, each particle moves about the center in a bound orbit. In the Newtonian limit, each orbit is a closed ellipse, and for static shells each orbit is circular (compare [20]). Since each particle follows a geodesic, circular orbits can be identified without ambiguity. These orbits can then be compared with those obtained from the turning-point and mass methods.

In the following we will focus on a moment of time symmetry, when at least momentarily each particle is in a purely tangential orbit $u^r = 0$ (where u^a is the four-velocity). The spherically symmetric line element can then be written as

$$ds^2 = -\alpha^2 dt^2 + \psi^4(dr^2 + r^2(d\theta + \sin^2\theta d\phi)), \qquad (1)$$

where α is the lapse function and ψ the conformal factor.

The rest mass M_0 of the shell can be computed from

$$M_0 = \int \rho_0 u^t \sqrt{-g} d^3x = 4\pi \int \rho_0 W \psi^6 r^2 dr, \qquad (2)$$

where g is the determinant of the spacetime metric and where we have defined the particles' Lorentz factor $W \equiv -\alpha u^t$. Since the shell's co-moving density ρ_0, which is a sum of the individual particle densities ρ_0^A, vanishes

everywhere except at the radius R of the shell, we find

$$\rho_0 = \sum_A \rho_0^A = \frac{M_0}{4\pi R^2 W \psi^6}\,\delta(r-R). \tag{3}$$

The conformal factor ψ in (1) can now be found from the Hamiltonian constraint

$$\nabla^2\psi = -2\pi\psi^5\rho_N, \tag{4}$$

and, following GGB, the lapse α from the maximal slicing condition

$$\nabla^2(\alpha\psi) = 2\pi\alpha\psi^5(\rho_N + 2S). \tag{5}$$

Here ρ_N is the density measured by a normal observer n^a

$$\rho_N = n^a n^b T_{ab} = n^a n^b \sum_A \rho_0^A u_a^A u_b^A = \rho_0 W^2, \tag{6}$$

(compare [9, 21]), and S is the trace of the spatial stress

$$S = \gamma^{ij}T_{ij} = \rho_0\gamma^{ij}u_i u_j = \rho_0(W^2-1), \tag{7}$$

where we have used the normalization condition

$$1 = W^2 - \gamma^{ij}u_i u_j. \tag{8}$$

For time symmetry both the momentum density $j^a = -\gamma^{ab}n^c T_{bc}$ and the extrinsic curvature vanish, so that a zero shift $\beta^i = 0$ identically satisfies the shift equation obtained in the conformal thin-sandwich decomposition.

The Hamiltonian constraint (4) and the maximal slicing condition (5) can readily be solved analytically by matching two vacuum solutions at the shell's radius R. Choosing the vacuum solutions such that the interior solution is regular at the center, while the exterior solution is regular at infinity, we find for the conformal factor

$$\psi = \begin{cases} 1 + \dfrac{W}{2\psi|_{\bar R}\bar R} & \text{for } \ 0 \le \bar r < \bar R \\ 1 + \dfrac{W}{2\psi|_{\bar R}\bar r} & \text{for } \ \bar r \ge \bar R. \end{cases} \tag{9}$$

Here and in the following we non-dimensionalize all quantities with respect to M_0, e.g. $\bar r \equiv r/M_0$. The value of $\psi|_{\bar R}$ can be found by evaluating the conformal factor at $\bar r = \bar R$, which yields a quadradic equation with the solution

$$\psi|_{\bar R} = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{W}{2\bar R}}. \tag{10}$$

The sign has been chosen so that ψ approaches the gravitational potential $\phi_{\rm Newt}$ in the Newtonian limit.

In terms of the ADM mass [17, 22]

$$\bar M_{\rm ADM} = -\frac{1}{2\pi M_0}\oint_\infty D^i\psi d^2S_i = \frac{W}{\psi|_{\bar R}}, \tag{11}$$

the exterior conformal factor (9) can be written

$$\psi = 1 + \frac{\bar M_{\rm ADM}}{2\bar r} \quad \text{for } \ \bar r \ge \bar R. \tag{12}$$

The maximal slicing condition (5) can be solved analogously to the Hamiltonian constraint, yielding

$$\alpha\psi = \begin{cases} 1 - \dfrac{\alpha|_{\bar R}(3W^2-2)}{2W\psi|_{\bar R}\bar R} & \text{for } \ 0 \le \bar r < \bar R \\ 1 - \dfrac{\alpha|_{\bar R}(3W^2-2)}{2W\psi|_{\bar R}\bar r} & \text{for } \ \bar r \ge \bar R. \end{cases} \tag{13}$$

Dividing by ψ, we find in the exterior

$$\alpha = \frac{-\alpha|_{\bar R}(3W^2-2) + 2W\psi|_{\bar R}\bar r}{W^2 + 2W\psi|_{\bar R}\bar r} \quad \text{for } \ \bar r \ge \bar R. \tag{14}$$

Evaluating this expression at $\bar r = \bar R$ determines the coefficient $\alpha|_{\bar R}$

$$\alpha|_{\bar R} = \left(1 + \frac{2W^2-1}{\psi|_{\bar R}\bar R W}\right)^{-1}. \tag{15}$$

Following GGB we now compute the Komar mass [18]

$$\bar M_{\rm K} = \frac{1}{4\pi M_0}\oint_\infty D^i\alpha\, d^2S_i = \frac{\alpha|_{\bar R}(3W^2-2) + W^2}{2W\psi|_{\bar R}}. \tag{16}$$

We note that the Komar mass is a slicing dependent quantity, and that this particular form results from having imposed maximal slicing. In terms of the Komar and ADM masses, the exterior lapse α can be written

$$\alpha = \frac{2\bar r - (2\bar M_{\rm K} - \bar M_{\rm ADM})}{2\bar r + \bar M_{\rm ADM}} \quad \text{for } \ \bar r \ge \bar R. \tag{17}$$

This expression reduces to the lapse as identified from the Schwarzschild metric in isotropic coordinates (see, e.g., exercise 31.7 in [23]) only if the two masses agree, $\bar M_{\rm K} = \bar M_{\rm ADM}$ (compare criterion (31) below).

So far, the shell's radius $\bar R$ and Lorentz factor W appear independently in the above equations. It is intuitively clear that searching for circular orbits will yield a relation between the particles' angular velocity and the gravitational field, and hence between $\bar R$ and W. Since our model consists of collisionless particles, circular orbits can be determined directly by solving the geodesic equations. Since all particles are identical, it is sufficient to evaluate the equation of motion for one particle, which we take to orbit in the equatorial plane. We therefore have $u^\theta = u^r = 0$, so that the normalization condition (8) yields a relation between u^ϕ and W

$$(u^\phi)^2 = \frac{W^2-1}{\psi^4|_R R^2}, \tag{18}$$

where we temporarily drop the bar notation. We now evaluate the geodesic equation,

$$\frac{du^a}{d\lambda} + \Gamma^a_{bc}u^b u^c = 0, \tag{19}$$

for $a = r$ to find a condition for the particles to remain in a purely tangential orbit ($du^r/d\lambda = 0$)

$$\Gamma^r_{tt}(u^t)^2 + \Gamma^r_{\phi\phi}(u^\phi)^2 = 0. \tag{20}$$

Combining this with (18) and $W = \alpha u^t$ gives

$$W^2 = \left(1 + \frac{\psi^4|_R R^2 \Gamma^r_{tt}}{\alpha^2 \Gamma^r_{\phi\phi}}\right)^{-1}. \tag{21}$$

When evaluating the Christoffel symbols, we must take into account the discontinuity in the first derivative of the metric coefficients at $r = R$. By averaging such a quantity over an extended shell and letting the thickness of the shell go to zero, we find that the derivative has to be replaced with

$$\psi_{,r} \to \frac{1}{2}(\psi_{,r}|_+ + \psi_{,r}|_-) = \frac{1}{2}\psi_{,r}|_+. \tag{22}$$

Using this rule for both ψ and α we find

$$\Gamma^r_{\phi\phi} = \frac{M_{\rm ADM}}{2}\left(1 + \frac{M_{\rm ADM}}{2R}\right)^{-1} - R \tag{23}$$

and

$$\Gamma^r_{tt} = \frac{M_{\rm K}}{2R^2}\left(1 + \frac{M_{\rm ADM}}{2R}\right)^{-6}\left(1 - \frac{M_{\rm K}}{R + M_{\rm ADM}/2}\right) \tag{24}$$

at $r = R$. Inserting these into eq. (21) yields

$$W^2 = \left(1 - \frac{M_{\rm K}}{2R - 2M_{\rm K} + M_{\rm ADM}}\right)^{-1}. \tag{25}$$

After some algebraic manipulation and dividing out the unphysical root $W = 0$, eq. (25) can be expanded into

$$4W^5 - 6\bar{R}W^4 - 4W^3 + 10\bar{R}W^2 + W - 4\bar{R} = 0, \tag{26}$$

where we have reintroduced the bar notation. This is the condition relating W and $\bar{R}$ for circular orbits. It is easy to show that this equation reduces to $\bar{\Omega}^2 = \bar{R}^{-3}/2$ in the Newtonian limit (with $\bar{R} \gg 1$, $v \ll 1$ and $W \simeq 1 + v^2/2 = 1 + \bar{R}^2\bar{\Omega}^2/2$).

For black holes, alternative criteria have to be used to identify circular orbits. In the following we will compare the turning-point method adopted by CB and the mass criterion adopted by GGB.

In the turning-point method, a circular orbit is identified by finding an extremum of the ADM mass (or equivalently the binding energy) at constant angular momentum $\bar{u}_\phi$

$$\left.\frac{d\bar{M}_{\rm ADM}}{d\bar{R}}\right|_{\bar{u}_\phi} = 0. \tag{27}$$

In a Newtonian context, this condition arises naturally from Hamilton's equations of motion. We start by differentiating the normalization condition, $(\bar{u}_\phi)^2 = \psi^4|_{\bar{R}} M_0^2 \bar{R}^2 (W^2 - 1)$, with respect to $\bar{R}$ to find

$$\frac{dW}{d\bar{R}} = \frac{-(W^2-1)(1+b)}{\bar{R}W(1+b) + 4W^2 - 2} \tag{28}$$

for sequences of constant angular momentum, where for convenience we have abbreviated $b = (1 + 2W/\bar{R})^{1/2}$. We now locate an extremum of the ADM mass (11) by setting its derivative with respect to $\bar{R}$ equal to zero

$$\frac{dW}{d\bar{R}}\left(\frac{W}{\bar{R}b(1+b)} - 1\right) = \frac{W^2}{\bar{R}^2 b(1+b)}. \tag{29}$$

Combining (28) and (29) then yields the condition

$$W^2 = \frac{-\bar{R}(W^2-1)(1+b)(W - \bar{R}b(1+b))}{\bar{R}W(1+b) + 4W^2 - 2}. \tag{30}$$

Inserting b and eliminating the unphysical root $W = -\bar{R}/2$, eq. (30) can be expanded identically into eq. (26).

In the mass method of GGB, the condition for circular orbits is obtained by equating the ADM and Komar mass (as obtained from maximal slicing)

$$\bar{M}_{\rm ADM} = \bar{M}_{\rm K}. \tag{31}$$

Inserting (11) and (16) yields, after some manipulation and elimination of the unphysical root $W = 0$, again the condition (26). Thus we have established that both criteria yield the correct condition for circular orbits in our model problem.

Since (31) only holds for stationary spacetimes, this criterion is closely related to a relativistic virial theorem. This relation is also evident from the expansions of the ADM and Komar masses to first order in $\epsilon \sim 1/\bar{R} \sim v^2$,

$$\bar{M}_{\rm ADM} \simeq 1 - \frac{1}{2\bar{R}} + \frac{1}{2}v^2 = 1 + \bar{U} + \bar{T} \tag{32}$$

and

$$\bar{M}_{\rm K} \simeq 1 - \frac{1}{\bar{R}} + \frac{3}{2}v^2 = 1 + 2\bar{U} + 3\bar{T}, \tag{33}$$

where U and T are the Newtonian potential and kinetic energies of the spherical shell. The two expansions (32) and (33) are equal only if the Newtonian virial theorem $T = -U/2$ holds.

For completeness, we evaluate the relativistic virial theorem in spherical symmetry as derived by [19]

$$\int \left(4\pi S - \frac{1}{\psi^4}\left(\left(\frac{d\ln\alpha}{dr}\right)^2 - \frac{1}{2}\left(\frac{d\ln\psi^2}{dr}\right)^2\right)\right)\psi^6 r^2 dr = 0. \tag{34}$$

Computing the above integral in terms of the Komar and ADM masses yields

$$\frac{(W^2-1)}{W} - \frac{2\bar{M}_{\rm K}^2}{\bar{M}_{\rm ADM} - 2\bar{M}_{\rm K} + 2\bar{R}} + \frac{\bar{M}_{\rm ADM}^2}{2\bar{R}} = 0, \tag{35}$$

which can again be brought into the form (26).

We now briefly discuss the physical implications of the condition (26). Solving for $\bar{R}$ we find

$$\bar{R} = \frac{4W^5 - 4W^3 + W}{6W^4 - 10W^2 + 4}. \tag{36}$$

To find a minimum value for the radius of our shell, we extremize the above equation with respect to W, which yields

$$(2W^2 - 1)(6W^6 - 21W^4 + 15W^2 - 2) = 0. \tag{37}$$

The only physical root (i.e. W real and $W \geq 1$) is $W = 1.607$, corresponding to $\bar{R}_{\rm min} = 1.532$. Expressing this in terms of $M_{\rm ADM}$ and circumferential radius $R_{\rm C}$ we find

$$\left(\frac{R_{\rm C}}{M_{\rm ADM}}\right)_{\rm min} = 2.506 \quad \text{(equilibrium)}. \tag{38}$$

This value should be compared with the Buchdahl limit $(R_{\rm C}/M_{\rm ADM})_{\rm min} = 9/4 = 2.25$ [24] for static fluid balls and $(R_{\rm C}/M_{\rm ADM})_{\rm min} = 3$ [23] for test particles in circular orbit in Schwarzschild spacetimes.

Requiring the particles' orbits to be stable leads to a more stringent limit on the compaction, which we find by requiring the second derivative of $\bar{M}_{\rm ADM}$ with respect to $\bar{R}$ to vanish in addition to (27). This yields an equation for W with the physical root $W = 1.108$ corresponding to $\bar{R}_{\rm min} = 3.053$, or

$$\left(\frac{R_{\rm C}}{M_{\rm ADM}}\right)_{\rm min} = 4.265 \quad \text{(stability)}, \tag{39}$$

which should be compared with the innermost stable circular orbit $(R_{\rm C}/M_{\rm ADM})_{\rm min} = 6$ of test particles in Schwarzschild spacetimes.

To summarize, we construct an analytic solution to Einstein's field equations describing a thin spherical shell consisting of collisionless particles in circular orbits. We apply the turning-point criterion (27) used by CB and the mass criterion (31) used by GGB and find that both conditions correctly identify circular orbits. The later criterion is intimately related to adopting maximal slicing, which is a natural choice for constructing quasi-equilibrium spacetimes (compare [16]). Our calculation illustrates these two criteria in the context of a very transparent, analytical framework and provides further evidence that the differences between the findings of CB and GGB result from the different initial value decompositions.

Acknowledgments

MLS gratefully acknowledges support through the Surdna Foundation Undergraduate Research Fellowship Program. We would also like to thank R. H. Price and K. S. Thorne for useful conversations, as well as the Visitors Program in Numerical Relativity at Caltech, where this project was initiated, for extending their hospitality. This work was supported in part by NSF Grant PHY 01-39907 to Bowdoin College.

[1] G. B. Cook, Phys. Rev. D **50**, 5025 (1994).
[2] T. W. Baumgarte, Phys. Rev. D **62**, 024018 (2000).
[3] H. Pfeiffer, S. A. Teukolsky and G. B. Cook, Phys. Rev. D **62**, 104018 (2000).
[4] E. Gourgoulhon, P. Grandclément and S. Bonazzola, Phys. Rev. D **65**, 044020 (2002); P. Grandclément, E. Gourgoulhon and S. Bonazzola, Phys. Rev. D **65**, 044021 (2002).
[5] B. D. Baker, submitted (also gr-qc/0205082).
[6] W. Tichy, B. Brügmann, M. Campanelli and P. Diener, submitted (also gr-qc/0207011).
[7] L. Blanchet, Phys. Rev. D **65**, 124009 (2002); T. Damour, E. Gourgoulhon and P. Grandclément, Phys. Rev. D **66**, 024007 (2002).
[8] In principle it is possible that the choices and approximations made in GGB and the post-Newtonian calculations (e.g. [9, 10]) lead to similar errors, so that their agreement might be misleading. Fully self-consistent numerical relativity evolution calculations would therefore provide a more reliable indication as to which data-sets indeed lead to circular orbits.
[9] T. Damour, P. Jaranowski and G. Schäfer, Phys. Rev. D **62**, 084011 (2000).
[10] A. Buonanno and T. Damour, Phys. Rev. D **59**, 084006 (1999).
[11] H. Pfeiffer, G. B. Cook and S. A. Teukolsky, Phys. Rev. D **66**, 024047 (2002).
[12] H. Pfeiffer and J. W. York, Jr., submitted (also gr-qc/0207095).
[13] Bowen and J. W. York, Jr., Phys. Rev. D **21**, 2047 (1980).
[14] J. Wilson and G. Mathews, Phys. Rev. Lett. **75**, 4161 (1995).
[15] J. W. York, Jr., Phys. Rev. Lett. **82**, 1350 (1999).
[16] G. B. Cook, Living Rev. Rel. **5**, 1 (2000).
[17] R. Arnowitt, S. Deser and C.W. Misner, *Gravitation* edited by L. Witten (New York: Wiley, 1962).
[18] A. Komar, Phys. Rev. **113**, 934 (1959).
[19] E. Gourgoulhon and S. Bonazzola, Class. Quantum Grav. **11**, 443 (1994).
[20] H.-J. Yo, T. W. Baumgarte and S. L. Shapiro, Phys. Rev. D **63** 064035 (2001).
[21] S. L. Shapiro and S. A. Teukolsky, Astrophys. J. **298**, 34 (1985); Phys. Rev. D **47**, 1529 (1993).
[22] N. Ó Murchadha and J. W. York, Jr., Phys. Rev. D **10**, 2345 (1974).
[23] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation* (New York: W. H. Freeman and Company, 1973).
[24] H. A. Buchdahl, Phys. Rev. **116**, 1027 (1959).